International Journal of Theoretical Physics, Vol. 29, No. 9, 1990

Energy and Momentum from the Palatini Formalism

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Received January 25, 1990

I derive from the Palatini formalism, in which metric and affinity are varied independently, an energy-momentum complex qualitatively different in form from the usual energy-momentum representations of general relativity. A similar procedure can be carried out for electrodynamics, illuminating by analogy the structure of the gravitational Lagrangian. The new energy density vanishes for all static vacuum solutions of the Einstein equations, and the radiated energy from an isolated system in an asymptotically flat space in general diverges. These facts suggest that the formalism could be used to express Mach's principle.

1. INTRODUCTION

There are many ways to describe the distribution of energy and momentum in general relativity. An infinity of "pseudotensors," "pseudotensor densities," and "complexes," here denoted by the general symbol t_μ^ν, have the property that, when combined with the material energy-momentum density $\mathfrak{T}_\mu^\nu$, they satisfy the conservation equation $\partial_\nu(\mathrm{t}_\mu^\nu + \mathfrak{T}_\mu^\nu) = 0$. In spite of its appearance, this is a covariant relationship, fully equivalent to $T_{\mu;\nu}^\nu = 0$. As Schrödinger (1950) put it, a "sham divergence" compensates for a "sham tensor." The "pseudo" character of t_μ^ν is an expression of the fact that energy and momentum cannot be localized in an invariant way, a consequence of the equivalence principle. Formally, it is due to the fact that t_μ^ν involves only $g_{\mu\nu}$ and its first derivatives, and the latter can be made to vanish at any space-time event by a coordinate transformation. The conservation laws do, however, allow a meaningful definition of the *total* energy and momentum of an isolated system via $P_\mu = \iiint (\mathrm{t}_\mu^\nu + \mathfrak{T}_\mu^\nu)\, d\sigma_\nu$.

The Lagrangian for general relativity can be written as $\mathfrak{L} = \mathfrak{L}_G + \mathfrak{L}_M$. The gravitational part $\mathfrak{L}_G$ is $\alpha \mathrm{g}^{\mu\nu} R_{\mu\nu}$, with $R_{\mu\nu}$ the Ricci tensor,

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\beta_{\mu\alpha}\Gamma^\alpha_{\nu\beta} - \Gamma^\beta_{\mu\nu}\Gamma^\alpha_{\beta\alpha}$$

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0020-7748/90/0900-1003$06.00/0

and $\alpha = 1/16\pi$. Here $\mathfrak{L}_M$ is the material Lagrangian, a function of $\mathfrak{g}^{\mu\nu}$, the nongravitational variables φ^A, and the latter's derivatives.

The oldest and most straightforward way of developing a gravitational energy-momentum pseudotensor begins by assuming that the $\Gamma^\alpha_{\mu\nu}$ are the Christoffel affinities. Second derivatives of $g_{\mu\nu}$ in $\mathfrak{L}_G$ are eliminated by isolating them in an ordinary divergence by means of the identity

$$\mathfrak{g}^{\mu\nu}R_{\mu\nu} \equiv (\mathfrak{g}^{\tau\rho}\Gamma^\alpha_{\tau\alpha} - \mathfrak{g}^{\tau\alpha}\Gamma^\rho_{\tau\alpha})_{,\rho} - \mathfrak{g}^{\tau\rho}(\Gamma^\sigma_{\tau\alpha}\Gamma^\alpha_{\rho\sigma} - \Gamma^\alpha_{\tau\rho}\Gamma^\sigma_{\alpha\sigma}) \tag{1}$$

The ordinary divergence may be dropped, and we can then write $\mathfrak{L}' = \mathfrak{L}'_G + \mathfrak{L}_M$, with

$$\mathfrak{L}'_G = -\alpha\mathfrak{g}^{\tau\rho}(\Gamma^\sigma_{\tau\alpha}\Gamma^\alpha_{\rho\sigma} - \Gamma^\alpha_{\tau\rho}\Gamma^\sigma_{\alpha\sigma}) \tag{2}$$

The Einstein (1916) pseudotensor may now be defined by the canonical prescription

$$\mathfrak{t}^\nu_\mu = (\partial\mathfrak{L}'_G/\partial g_{\alpha\beta,\nu})g_{\alpha\beta,\mu} - \delta^\nu_\mu\mathfrak{L}'_G \tag{3}$$

(From now on $\mathfrak{t}^\nu_\mu$ will denote this object. We will not deal with any of the other conventional pseudotensors, though for some purposes they are more useful than the Einstein expression. In common with it, they involve only $g_{\mu\nu}$ and $g_{\mu\nu,\sigma}$ and are quadratic in the first derivatives. Thus, all these objects are analogous to the usual energy-momentum tensor for electrodynamics.) Explicitly,

$$\mathfrak{t}^\nu_\mu = \alpha[\mathfrak{g}^{\alpha\beta}_{,\mu}\Gamma^\nu_{\alpha\beta} - \mathfrak{g}^{\alpha\nu}_{,\mu}\Gamma^\beta_{\alpha\beta} + \delta^\nu_\mu\mathfrak{g}^{\tau\rho}(\Gamma^\sigma_{\tau\alpha}\Gamma^\alpha_{\rho\sigma} - \Gamma^\alpha_{\tau\rho}\Gamma^\sigma_{\alpha\sigma})] \tag{4}$$

Now there is another approach to the dynamics of general relativity which is well known, that of Palatini (1919). However, the quite different expression for gravitational energy and momentum to which the Palatini formalism leads does not seem to have been used. My purpose here is to discuss this object and its possible significance.

We return to our original Lagrangian, and no longer assume $\Gamma^\sigma_{\tau\rho}$ to be the Christoffel affiinity $\{^{\sigma}_{\tau\rho}\}$. Instead, $\Gamma^\sigma_{\tau\rho}$ and $\mathfrak{g}^{\alpha\beta}$ are treated as independent fields for Hamilton's principle. The Lagrangian will then have the form

$$\mathfrak{L} = \alpha\mathfrak{g}^{\mu\nu}R_{\mu\nu}(\Gamma, \partial\Gamma) + \mathfrak{L}_M(\varphi, \partial\varphi, \mathfrak{g}) \tag{5}$$

Variation of $\Gamma^\sigma_{\tau\rho}$ yields $\Gamma^\alpha_{\mu\nu} = \{^{\alpha}_{\mu\nu}\}$ as a *consequence*, variation of $\mathfrak{g}^{\alpha\beta}$ gives the Einstein equations, and varying φ^A leads to the equations of motion for the nongravitational variables.

All of this is familiar. But we may go beyond the derivation of the field equations and calculate the canonical energy-momentum expression in the Palatini formalism,

$$\mathfrak{K}^\nu_\mu = (\partial\mathfrak{L}_G/\partial\Gamma^\alpha_{\beta\sigma,\nu})\Gamma^\alpha_{\beta\sigma,\mu} - \delta^\nu_\mu\mathfrak{L}_G \tag{6}$$

This corresponds to (3) in Einstein's approach. Equation (6) has no term involving derivatives of $\mathfrak{L}_G$ with respect to $\mathfrak{g}^{\alpha\beta}_{,\nu}$, since the Lagrangian contains no derivatives of the metric. Similarly, the canonical energy-momentum tensor for the nongravitational fields is

$$\mathfrak{S}^{\nu}_{\mu} = (\partial \mathfrak{L}_M / \partial \varphi^A_{,\nu}) \varphi^A_{,\mu} - \delta^{\nu}_{\mu} \mathfrak{L}_M \tag{7}$$

It follows from the general Lagrangian formalism that the total canonical energy-momentum density will satisfy the conservation law $\partial_\nu(\mathfrak{R}^{\nu}_{\mu} + \mathfrak{S}^{\nu}_{\mu}) = 0$.

The object $\mathfrak{R}^{\nu}_{\mu}$ has been encountered previously in nonsymmetric unified field theories (Einstein, 1955, Appendix 2; Murphy, 1976). In a purely affine theory, this is the only canonical energy-momentum expression which can be constructed. Here we wish to investigate this object within the context of Einstein's theory of gravitation.

2. LAGRANGIANS FOR ELECTRODYNAMICS

Before we consider the properties of gravitational energy and momentum given by the Palatini formalism, it will be instructive to examine the corresponding procedure for Maxwell's theory. This gives new ways of understanding both electrodynamics and gravitation. For electrodynamics we may use the Lagrangian

$$\mathfrak{L}_1 = \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{8}$$

with $F_{\mu\nu}$ merely an abbreviation for $A_{\nu,\mu} - A_{\mu,\nu}$. Varying the vector potential A_μ then gives the field equations $F^{\mu\nu}_{,\nu} = 0$, while the set $F_{[\mu\nu,\rho]} = 0$ is an automatic consequence of the definition of $F_{\mu\nu}$ in terms of A_μ. The canonical energy-momentum tensor is

$$E^{\nu}_{1\mu} = (\partial \mathfrak{L}_1 / \partial A_{\sigma,\nu}) A_{\sigma,\mu} - \delta^{\nu}_{\mu} \mathfrak{L}_1 = -A_{\sigma,\mu} F^{\sigma\nu} - \tfrac{1}{4} \delta^{\nu}_{\mu} F_{\alpha\rho} F^{\alpha\rho} \tag{9}$$

which differs from the symmetric expression $F_{\sigma\mu} F^{\sigma\nu} - \frac{1}{4} \delta^{\nu}_{\mu} F_{\alpha\beta} F^{\alpha\beta}$ by the divergence-free quantity $A_{\mu,\sigma} F^{\sigma\nu}$.

But we may also write the Lagrangian in the form

$$\mathfrak{L}_2 = -(A_{\mu,\nu} F^{\mu\nu} + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}) \tag{10}$$

with no *a priori* relation between A_μ and $F_{\mu\nu}$ (Arnowitt *et al.*, 1962). [However, (10) reduces to (8) when the usual relation is assumed.] This is the analogue of the Palatini approach in general relativity. Varying $F^{\mu\nu}$ now yields $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, while variation of A_μ results in $F^{\sigma\tau}_{,\tau} = 0$. The canonical energy-momentum tensor calculated from $\mathfrak{L}_2$ is the same as that from $\mathfrak{L}_1$.

We may add the divergence $(A_\mu F^{\mu\nu})_{,\nu}$ to $\mathfrak{L}_2$ to obtain

$$\mathfrak{L}_3 = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu F^{\mu\nu}_{,\nu} \tag{11}$$

Now A_μ plays the role of a set of Lagrange multipliers for *constraints* $F^{\mu\nu}_{,\nu}=0$. Variation of $F_{\mu\nu}$, taking account of its antisymmetry, will again give $F_{\mu\nu}=A_{\nu,\mu}-A_{\mu,\nu}$. The canonical energy-momentum tensor is now something new, involving derivatives of the field strengths:

$$E_{3\mu}^{\nu}=A_\sigma F^{\sigma\nu}_{,\mu}+\tfrac{1}{4}\delta^{\nu}_{\mu}F_{\alpha\beta}F^{\alpha\beta} \tag{12}$$

It is interesting that what are usually regarded as the dynamically significant field equations, $F^{\mu\nu}_{,\nu}=0$, appear here as a set of constraints. We may compare this with the gravitational situation. The vacuum Lagrangian $\mathfrak{L}_G=\alpha\mathfrak{g}^{\mu\nu}R_{\mu\nu}$ is seen to have the form of a set of pure constraints $R_{\mu\nu}=0$, the vacuum Einstein equations, with $\mathfrak{g}^{\mu\nu}$ the corresponding set of Lagrange multipliers. This gives us at the outset a novel view of the dynamics of general relativity.

3. THE CANONICAL PSEUDOTENSOR FOR THE PALATINI ACTION

The gravitational energy-momentum pseudotensor defined by (6) is easily found to be

$$\mathfrak{K}^{\nu}_{\mu}=\alpha(\mathfrak{g}^{\alpha\nu}\Gamma^{\sigma}_{\alpha\sigma,\nu}-\mathfrak{g}^{\alpha\sigma}\Gamma^{\nu}_{\alpha\sigma,\mu}-\delta^{\nu}_{\mu}\mathfrak{R}) \tag{13}$$

For pure gravitation $\mathfrak{R}=0$, and $\mathfrak{K}^{\nu}_{\mu}$, which now represents the total energy and momentum density, reduces to the simple expression

$$\mathfrak{K}^{\nu}_{\mu}=2\alpha\mathfrak{g}^{\alpha[\nu}\Gamma^{\sigma]}_{\alpha\sigma,\mu} \tag{14}$$

From this we immediately obtain an important result which was pointed out by Einstein (1955, Appendix 2) in the context of his nonsymmetric theory. For a static vacuum space-time there will be frames in which the affinity is independent of time, so that $\mathfrak{K}^{\nu}_{0}=0$. Thus, for such a space-time there will always be a frame in which there is no total energy or momentum.

When the Christoffel affinity is substituted in (13) in accord with the field equations, we find that $\mathfrak{K}^{\nu}_{\mu}$ is linear in the second derivatives of the metric. Thus, it differs both from the conventional pseudotensors and from intuitive ideas about energy density and flux. $\mathfrak{K}^{\nu}_{\mu}$ has this feature in common with an object proportional to the Einstein tensor which was proposed by Lorentz (1916; see also Pauli, 1958) as a gravitational energy-momentum tensor. But, unlike Lorentz's object, $\mathfrak{K}^{\nu}_{\mu}$ does not yield identically vanishing densities for total energy and momentum.

One naturally asks about the relationship between $\mathfrak{K}^{\nu}_{\mu}$ and t^{ν}_{μ}. If the identity (1) is used to transform $\mathfrak{R}$ in (13), and $\mathfrak{g}^{\mu\nu}_{,\sigma}$ is written in terms of $\mathfrak{g}^{\mu\nu}$ and $\{{}^{\sigma}_{\tau\rho}\}$, we obtain

$$\mathfrak{K}^{\nu}_{\mu}=\mathfrak{t}^{\nu}_{\mu}+(\mathfrak{A}^{\nu}_{,\mu}-\delta^{\nu}_{\mu}\mathfrak{A}^{\rho}_{,\rho}) \tag{15}$$

where

$$\mathfrak{A}^{\mu} \equiv \alpha[\mathfrak{g}^{\nu\mu}\{^{\sigma}_{\nu\sigma}\} - \mathfrak{g}^{\nu\sigma}\{^{\mu}_{\nu\sigma}\}] \tag{16}$$

It is interesting that the term added to $\mathfrak{t}^{\nu}_{\mu}$ in (15) in fact has a vanishing ordinary divergence for *any* $\mathfrak{A}^{\mu}$: $\mathfrak{A}^{\nu}_{,u\nu} - \delta^{\nu}_{\mu}\mathfrak{A}^{\rho}_{,\rho\nu} \equiv 0$. Thus, the use of (15) with an arbitrary $\mathfrak{A}^{\mu}$ will give conservation laws. But (16), which arises directly from the Palatini action, will be of special interest.

We also note that the particular form of $\mathfrak{K}^{\nu}_{\mu}$ means that the difference between the energy densities calculated from $\mathfrak{K}^{\nu}_{\mu}$ and $\mathfrak{t}^{\nu}_{\mu}$ will always be an ordinary spatial divergence: $\mathfrak{K}^{0}_{0} - \mathfrak{t}^{0}_{0} = -\mathfrak{A}^{m}_{,m}$. In an asymptotically flat space, the energies calculated from $\mathfrak{K}^{0}_{0}$ and $\mathfrak{t}^{0}_{0}$ will differ by a surface integral.

The canonical formalism tells us that $\partial_{\nu}(\mathfrak{K}^{\nu}_{\mu} + \mathfrak{S}^{\nu}_{\mu}) = 0$, where $\mathfrak{S}^{\nu}_{\mu}$ is given by (7). $\mathfrak{S}^{\nu}_{\mu}$ may differ from the material energy-momentum tensor density more often used in general relativity, $\mathfrak{T}^{\nu}_{\mu} = 2g_{\sigma\mu}\delta\mathfrak{L}_M/\delta g_{\nu\sigma}$, by a quantity whose divergence vanishes.

It can be shown (Møller, 1972; see also Weyl, 1922) that the sum of $\mathfrak{T}^{\nu}_{\mu}$ and the Einstein pseudotensor may in general be written as a divergence: $\mathfrak{t}^{\nu}_{\mu} + \mathfrak{T}^{\nu}_{\mu} = \mathfrak{s}^{\nu\rho}_{\mu,\rho}$, where $\mathfrak{s}^{\nu\rho}_{\mu} = 2(\partial\mathfrak{L}'_G/\partial g^{\mu\tau}_{,\nu})g^{\nu\tau}$. For a static vacuum solution this yields $\mathfrak{t}^{0}_{0} = \frac{1}{2}\mathfrak{s}^{\mu i}_{\mu,i}$. Explicit calculation gives

$$\mathfrak{t}^{0}_{0} = \alpha(\mathfrak{g}^{\rho m}\Gamma^{\tau}_{\rho\tau} - \mathfrak{g}^{\rho\tau}\Gamma^{m}_{\rho\tau})_{,m} = \mathfrak{A}^{m}_{,m}$$

This is precisely canceled in $\mathfrak{K}^{0}_{0}$ by the $-\mathfrak{A}^{m}_{,m}$ term, confirming our general result that $\mathfrak{K}^{0}_{0}$ vanishes for a static vacuum space-time. But it is important to note that *all* the components of $\mathfrak{K}^{\nu}_{\mu}$ do not vanish. In particular, for the Schwarzschild solution in the common coordinates we find $\mathfrak{K}^{r}_{\theta} = 4m\cos\theta$.

We may also calculate $\mathfrak{K}^{\nu}_{\mu}$ for exact solutions of the vacuum equations without a timelike Killing vector. With both $\mathfrak{t}^{\nu}_{\mu}$ and $\mathfrak{K}^{\nu}_{\mu}$, the lack of a proper tensorial character makes interpretation of the results somewhat uncertain. For example, we may consider a gravitational plane wave described first by the metric

$$ds^2 = A\,du^2 + du\,dv - dx^2 - dy^2$$

where $u = t - z$, $v = t + z$, and $A = G(u)(x^2 - y^2)$, G being an arbitrary function (Sachs, 1967). It is easy to show that all the components of both $\mathfrak{t}^{\nu}_{\mu}$ and $\mathfrak{K}^{\nu}_{\mu}$ vanish in these coordinates. On the other hand, we may make a coordinate transformation and write the metric as

$$ds^2 = dt^2 - L^2(e^{2\beta}\,dx^2 + e^{-2\beta}\,dy^2) - dz^2$$

with L and β functions of $u = t - z$. With this form we obtain $\mathfrak{t}^{3}_{0} = 4\alpha L^2(\beta'^2 - L'^2/L^2)$. In the weak field limit, $L \to 1$ and $\mathfrak{t}^{3}_{0} \to 4\alpha\beta'^2$, which one might expect from an analogy with the electromagnetic Poynting vector. The Palatini formalism, however, gives $\mathfrak{K}^{3}_{0} = 4\alpha L^2(\beta'^2 + L''/L)$, and this vanishes because the single nontrivial Einstein equation is precisely $L'' + L\beta'^2 = 0$.

A more disconcerting result is obtained when we consider radiation from a bounded source. When metric perturbations have the asymptotic form $h_{\mu\nu} = a_{\mu\nu} r^{-1} \exp[i\omega(r-t)]$ with constant $a_{\mu\nu}$'s so that the usual radiation condition is satisfied, then t_μ^ν, which is quadratic in first derivatives of the metric, will contain terms with a $1/r^2$ dependence. Calculation of the total energy flux yields the standard result $\dot{E} = -\dddot{Q}_{lm} \dddot{Q}^{lm}/45$ for the power radiated by a system whose quadruple tensor is Q_{lm} (Møller, 1972). But the situation is quite different when $\mathfrak{K}_\mu^\nu$ is used. In the weak-field approximation, this object has terms which are proportional to second derivatives of the metric perturbation. Thus, the energy flux in the "radiation zone" would be proportional to $1/r$ rather than $1/r^2$, and the integrated flux will diverge unless the coefficient of the $1/r$ term happens to vanish. This result seems to make $\mathfrak{K}_\mu^\nu$ useless for calculations of radiated power from finite systems. But this object may be significant in a cosmological context. We proceed to discuss that possibility.

4. MACH'S PRINCIPLE

The Palatini formalism for energy and momentum produces some unusual results, as we have seen. The mass of an isolated particle (Schwarzschild or Kerr metric) vanishes, and radiation from an isolated system cannot be given a simple meaning. These facts might tempt one to believe that $\mathfrak{K}_\mu^\nu$ can be of little use.

However, there is an oft-discussed but still inadequately formulated concept in connection with which an object such as $\mathfrak{K}_\mu^\nu$ might be of interest. Mach's principle is the idea that in some way the inertia of any body is due to its interaction with the rest of the universe, and Einstein began the attempts to derive this result from general relativity by trying to show how inertial effects could arise through the field equations and the equations of motion (Mach, 1893; Einstein, 1955, pp. 99–108; Sciama, 1959; Murphy, 1976).

But the fact that general relativity allows solutions such as that of Schwarzschild, in which a single isolated particle has mass, means that Mach's principle is violated in a basic way in this theory. This is also true in other field theories in which a single particle has a nonvanishing self-energy and thus—because of Einstein's $E = mc^2$—a nonzero self-*mass*. The "self-energy problem" in field theories has usually referred to a divergence of the self-energy of a particle. But from the standpoint of Mach's principle, any nonzero self-energy for a single particle is a problem. While this mass arises from interaction with a field, it is strongly localized in the vicinity of the particle in question. (For example, in quantum electrodynamics, most of the energy of an electron is concentrated in a region with dimensions on the order of $(\hbar/mc)\exp(-\hbar c/e^2)$ (Weisskopf, 1939). In general relativity the energy cannot be localized in a way that has any invariant significance, but the fact that there is any energy at all is a problem.

The use of $\mathfrak{R}_\mu^\nu$ to represent energy and momentum would begin to overcome this difficulty, since with this object there is no mass for an isolated particle in a proper coordinate system. (This result is due in part to our focus on vacuum solutions, since there is then no material energy-momentum tensor.) Of course, this is only a first step toward an adequate formulation of Mach's principle. One would like to show how inertia of a particle actually arises from its interactions with the rest of the matter in the universe. But there is little point in trying to take that step until self-inertia has been eliminated.

The divergence of the radiative flux from an isolated system also seems to point to the need to take Mach's principle into account more fully. The usual treatment of radiation assumes an asymptotically flat space, but a universe in accord with Mach's principle probably would not allow such a solution. Wheeler (1962) has argued in the past for an interpretation of Mach's principle as a boundary condition for the Einstein equations, a condition that space be "properly closed." The radiative divergence which we have noted suggests that such a condition is indeed necessary.

REFERENCES

Arnowitt, R., Deser, S., and Misner, C. W. (1962). In *Gravitation: An Introduction to Current Research*, L. Witten, ed., Wiley, New York, Chapter 7.

Einstein, A. (1916). *Sitzungberichte der Preussiche Akademie der Wisseschaften*, **1916**, 1115 [Reprinted in A. Einstein *et al., The Principle of Relativity*, Dover, New York].

Einstein, A. (1955). *The Meaning of Relativity*, 5th ed., Princeton University Press, Princeton, New Jersey.

Lorentz, H. A. (1916). *Versl. gewone. Vergab. Akad. Amst.*, **25**, 468, 1380.

Mach, E. (1893). *The Science of Mechanics*, 2d English ed., Open Court, Chicago, pp. 226–232.

Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco, Chapter 35.

Møller, C. (1972). *The Theory of Relativity*, 2d ed., Oxford University Press, Oxford, pp. 459–474.

Murphy, G. L. (1976). *Australian Journal of Physics*, **29**, 469.

Palatini, A. (1919). *Rend. Circ. Mat. Palermo*, **43**, 203.

Pauli, W. (1958). *Theory of Relativity*, Pergamon, New York, pp. 176–177.

Sachs, R. K. (1967). Gravitational waves, in *Relativity Theory and Astrophysics*, J. Ehlers, ed., American Mathematical Society, Providence, Rhode Island.

Schrödinger, E. (1950). *Space-Time Structure*, Cambridge University Press, Cambridge, Chapter XI.

Sciama, D. W. (1959). *The Unity of the Universe*, Doubleday, Garden City, New York.

Weisskopf, V. S. (1939). *Physical Review*, **56**, 72.

Weyl, H. (1922). *Space-Time-Matter*, Dover, New York, Section 33.

Wheeler, J. A. (1962). *Geometrodynamics*, Academic Press, New York, pp. 58–63.